1 - 10 Type and stability of critical point

Determine the type and stability of the critical point. Then find a real general solution and sketch or graph some of the trajectories in the phase plane.

I'm going to need to bring Tables 4.1

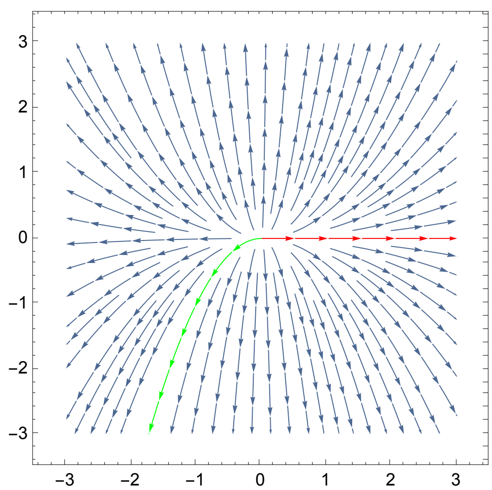
Name	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$	$\Delta=(\lambda_1-\lambda_2)^2$	Comments on λ_1, λ_2
(a) Node		$q>0$	$\Delta\geq 0$	Real, same sign
(b) Saddle point		$q<0$		Real, opposite signs
(c) Center	$p=0$	$q>0$		Pure imaginary
(d) Spiral point	$p\neq 0$		$\Delta<0$	Complex, not pure imaginary

and 4.2 in here for consultation.

Type of Stability	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$
(a) Stable and attractive	$p<0$	$q>0$
(b) Stable	$p\leq 0$	$q>0$
(c) Unstable	$p>0$ OR	OR $q<0$

1. $y_1' = y_1$
$y_2' = 2 y_2$

```
StreamPlot[{y1, 2 y2}, {y1, -3, 3}, {y2, -3, 3}, StreamPoints →
  {{{{1, 0}, Red}, {{-1, -1}, Green}, Automatic}}, ImageSize → 250]
```




```
ClearAll["Global`*"]
```

```
e1 = {y1'[t] == y1[t], y2'[t] == 2 y2[t]}
e2 = DSolve[e1, {y1, y2}, t]
```

{y1′[t] == y1[t], y2′[t] == 2 y2[t]}

$\{\{y1 \to \text{Function}[\{t\}, e^t C[1]], y2 \to \text{Function}[\{t\}, e^{2t} C[2]]\}\}$

1. Above: the general, real sol'ns.

```
te = e2[[1, 1, 2, 2]]
```

e^t C[1]

The solution for y1, below, matches the text.

```
fe = te /. C[1] → c1
```

c1 e^t

```
e3 = Eigensystem[{{1, 0}, {0, 2}}]
{{2, 1}, {{0, 1}, {1, 0}}}
```

$\lambda_1 = 2$

2

$\lambda_2 = 1$

1

$p = \lambda_1 + \lambda_2$

3

$q = \lambda_1 \lambda_2$

2

$\Delta = (\lambda_1 - \lambda_2)^2$

1

1. Because p>0, the critical point is unstable according to Table 4-2.

```
TableForm[Table[{t, c1, fe}, {t, 4}, {c1, -1, 1}],
 TableHeadings → {{}, {"t", "c1 ", "fe "}}]
```

t	c1	fe
1 -1 $-e$	1 0 0	1 1 e
2 -1 $-e^2$	2 0 0	2 1 e^2
3 -1 $-e^3$	3 0 0	3 1 e^3
4 -1 $-e^4$	4 0 0	4 1 e^4

```
fifo = Table[{t, fe}, {t, 4}, {c1, -1, 1}]
```

$\{\{\{1, -e\}, \{1, 0\}, \{1, e\}\}, \{\{2, -e^2\}, \{2, 0\}, \{2, e^2\}\}, \{\{3, -e^3\}, \{3, 0\}, \{3, e^3\}\}, \{\{4, -e^4\}, \{4, 0\}, \{4, e^4\}\}\}$

```
hiu[c1_, t_] := fe

plot1 = Plot[Evaluate[Table[hiu[c1, t], {c1, -8, 8}]],
   {t, -3, 3}, PlotRange → {-50, 50}, PlotStyle → Thickness[0.003]];
```

3. Above: This is a plot of the first sol'n, with trajectories of various constant values.

```
f[c1_, t_] := c1 e^t

VectorPlot[{1, f[c1, t]}, {c1, -3, 3}, {t, -1, 1},
  Axes → True, Frame → False, VectorScale → {Tiny, Tiny, None},
  BaseStyle → AbsoluteThickness[0.4`], PlotTheme → None, ImageSize → 250];

plot2 = VectorPlot[{1, f[t, c1]}, {c1, -3, 3}, {t, -8, 8},
   Axes → True, Frame → False, VectorScale → {Tiny, Tiny, None},
   BaseStyle → AbsoluteThickness[0.4], PlotTheme → None, ImageSize → 350];

Show[plot1, plot2];

fi = e2[[1, 2, 2, 2]]
```

$e^{2t}\, C[2]$

The solution for y2, below, agrees with the text.

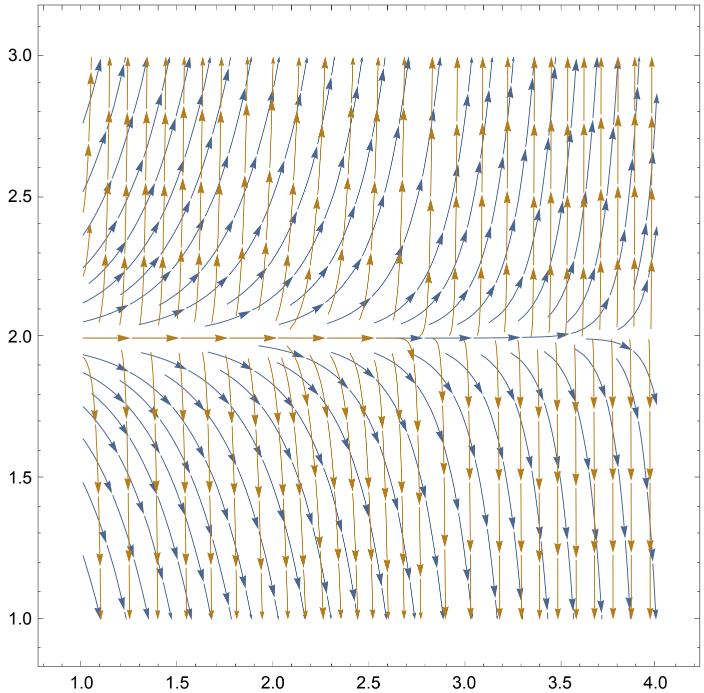
```
fif = fi /. C[2] → c2
```

$c2\, e^{2t}$

```
fifi = Table[{t, fif}, {t, 4}, {c2, -1, 1}]
```

$\{\{\{1, -e^2\}, \{1, 0\}, \{1, e^2\}\}, \{\{2, -e^4\}, \{2, 0\}, \{2, e^4\}\}, \{\{3, -e^6\}, \{3, 0\}, \{3, e^6\}\}, \{\{4, -e^8\}, \{4, 0\}, \{4, e^8\}\}\}$

```
ListStreamPlot[{fifo, fifi}]
```



3. $y_1' = y_2$
$y_2' = -9\, y_1$

```
ClearAll["Global`*"]
```

```
e1 = {y1'[t] == y2[t], y2'[t] == -9 y1[t]}
e2 = DSolve[e1, {y1, y2}, t]
```

{y1′[t] == y2[t], y2′[t] == -9 y1[t]}

$$\left\{\left\{y1 \to \text{Function}\left[\{t\}, C[1] \text{Cos}[3\,t] + \frac{1}{3} C[2] \text{Sin}[3\,t]\right], y2 \to \text{Function}[\{t\}, C[2] \text{Cos}[3\,t] - 3\,C[1] \text{Sin}[3\,t]]\right\}\right\}$$

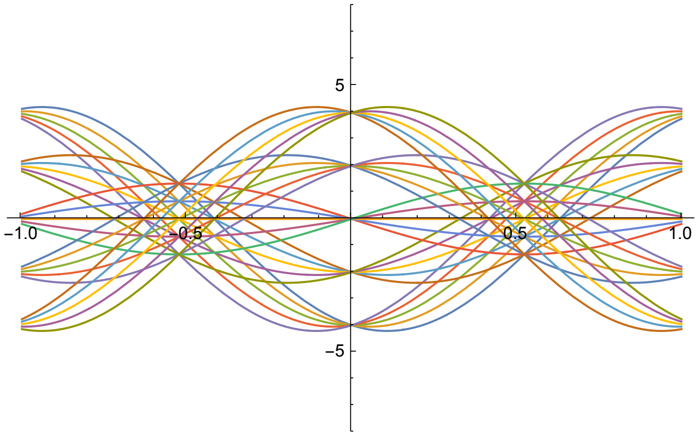
```
e3 = e2[[1, 1, 2, 2]]
```

$$C[1] \text{Cos}[3\,t] + \frac{1}{3} C[2] \text{Sin}[3\,t]$$

The solution for y_1, below, agrees with the text, provided that text constant A is assigned the value of C[1], and text constant B is assigned the value of $\frac{1}{3}$C[2].

$$\text{hiy}[c1\_, c2\_, t\_] := c1 \text{Cos}[3\,t] + \frac{1}{3} c2 \text{Sin}[3\,t]$$

```
plot1 =
 Plot[Evaluate[Table[hiy[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],
  {t, -1, 1}, PlotRange → {-8, 8}, PlotStyle → Thickness[0.003]]
```



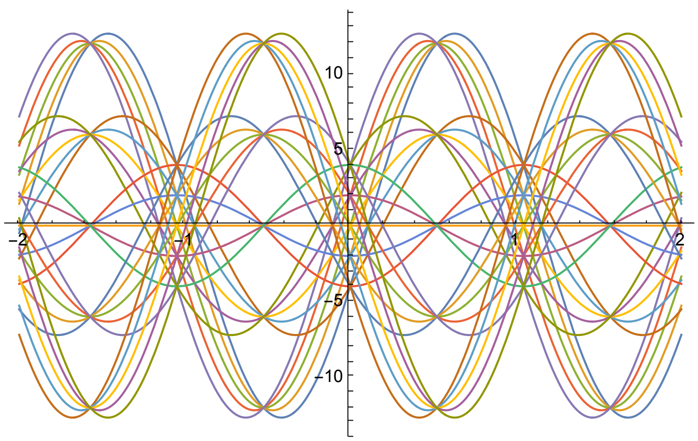
1. Above: Some trajectories of the first sol'n. Below: the solution for y_2 agrees with the text, with appropriate constant assignments.

```
e4 = e2[[1, 2, 2, 2]]
```

```
C[2] Cos[3 t] - 3 C[1] Sin[3 t]
```

```
hiz[c1_, c2_, t_] := c2 Cos[3 t] - 3 c1 Sin[3 t]
```

```
plot1 =
 Plot[Evaluate[Table[hiz[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],
  {t, -2, 2}, PlotRange → Automatic, PlotStyle → Thickness[0.003]]
```



2. Above: Some trajectories of the second sol'n.

```
e5 = Eigensystem[{{0, 1}, {-9, 0}}]
{{3 ⅈ, -3 ⅈ}, {{-ⅈ, 3}, {ⅈ, 3}}}
```

```
p = 3 ⅈ - 3 ⅈ
```

```
0
```

```
q = 3 ⅈ (-3 ⅈ)
```

```
9
```

```
Δ = (3 ⅈ - (-3 ⅈ))^2
```

```
-36
```

3. The system's critical point is center. According to Table 4-2, it is stable.

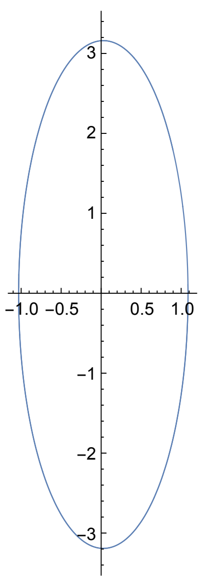
```
e3p = e3 /. {C[1] → 1, C[2] → 1}
```

$\mathrm{Cos}[3\,t] + \frac{1}{3}\,\mathrm{Sin}[3\,t]$

```
e4p = e4 /. {C[1] → 1, C[2] → 1}
```

```
Cos[3 t] - 3 Sin[3 t]
```

```
ParametricPlot[{e3p, e4p}, {t, -2, 2},
 ImageSize → 100, PlotStyle → Thickness[0.006]]
```



5. $y_1' = -2\,y_1 + 2\,y_2$
$y_2' = -2\,y_1 - 2\,y_2$

```
ClearAll["Global`*"]
```

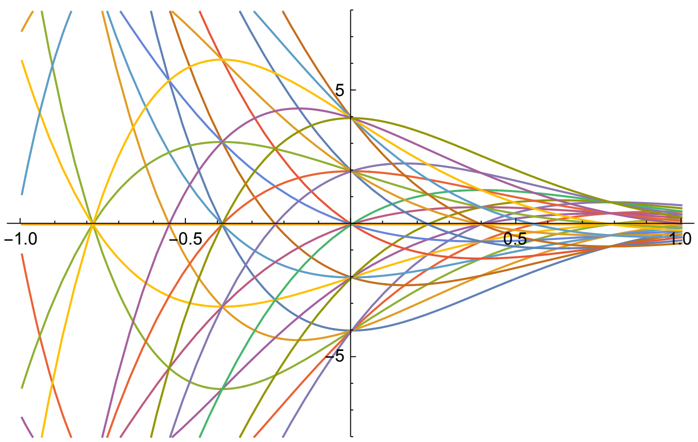
```
e1 = {y1'[t] == -2 y1[t] + 2 y2[t], y2'[t] == -2 y1[t] - 2 y2[t]}
e2 = DSolve[e1, {y1, y2}, t]
{y1′[t] == -2 y1[t] + 2 y2[t], y2′[t] == -2 y1[t] - 2 y2[t]}
{{y1 → Function[{t}, e^(-2 t) C[1] Cos[2 t] + e^(-2 t) C[2] Sin[2 t]],
  y2 → Function[{t}, e^(-2 t) C[2] Cos[2 t] - e^(-2 t) C[1] Sin[2 t]]}}
```

```
e3 = e2[[1, 1, 2, 2]]
e^(-2 t) C[1] Cos[2 t] + e^(-2 t) C[2] Sin[2 t]
```

```
hiy[c1_, c2_, t_] := e^(-2 t) c1 Cos[2 t] + e^(-2 t) c2 Sin[2 t]
```

Above: The green cell matches the answer in the text for y_1, assuming appropriate assignment of constants.

```
plot1 =
 Plot[Evaluate[Table[hiy[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],
  {t, -1, 1}, PlotRange → {-8, 8}, PlotStyle → Thickness[0.003]]
```

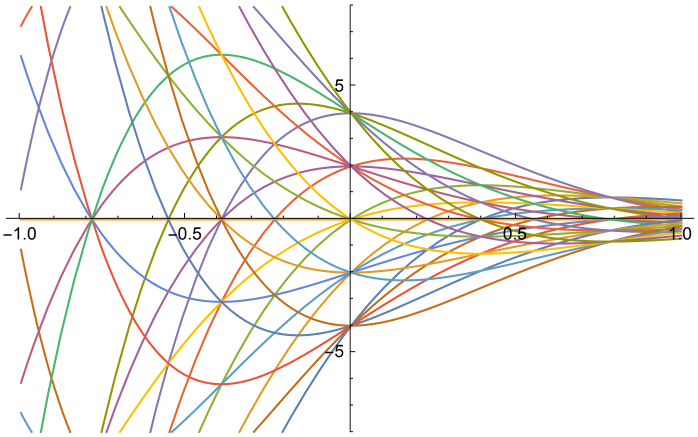



```
e4 = e2[[1, 2, 2, 2]]
e^(-2 t) C[2] Cos[2 t] - e^(-2 t) C[1] Sin[2 t]
```

```
hiz[c1_, c2_, t_] := e^(-2 t) c2 Cos[2 t] - e^(-2 t) c1 Sin[2 t]
```

Above: The green cell matches the answer in the text for y_2, assuming appropriate assignment of constants.

```
plot2 =
 Plot[Evaluate[Table[hiz[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],
  {t, -1, 1}, PlotRange → {-8, 8}, PlotStyle → Thickness[0.003]]
```




```
e5 = Eigensystem[{{-2, 2}, {-2, -2}}]
{{-2 + 2 ⅈ, -2 - 2 ⅈ}, {{-ⅈ, 1}, {ⅈ, 1}}}
```

```
p = -2 + 2 ⅈ + (-2 - 2 ⅈ)
```

```
-4
```

```
q = -2 + 2 ⅈ (-2 - 2 ⅈ)
```

```
2 - 4 ⅈ
```

```
Δ = ((-2 + 2 ⅈ) - (-2 - 2 ⅈ))^2
```

```
-16
```

According to Table 4-1, the critical point is a spiral point. If the presence of imaginary part does not matter, the point is stable.

7. $y_1' = y_1 + 2\,y_2$
$y_2' = 2\,y_1 + y_2$

```
ClearAll["Global`*"]
```

```
e1 = {y1'[t] == y1[t] + 2 y2[t], y2'[t] == 2 y1[t] + y2[t]}
e2 = DSolve[e1, {y1, y2}, t]
{y1′[t] == y1[t] + 2 y2[t], y2′[t] == 2 y1[t] + y2[t]}
```

$$\left\{\left\{y1 \to \text{Function}\left[\{t\}, \frac{1}{2} e^{-t} \left(1 + e^{4 t}\right) C[1] + \frac{1}{2} e^{-t} \left(-1 + e^{4 t}\right) C[2]\right],\right.\right.$$
$$\left.\left. y2 \to \text{Function}\left[\{t\}, \frac{1}{2} e^{-t} \left(-1 + e^{4 t}\right) C[1] + \frac{1}{2} e^{-t} \left(1 + e^{4 t}\right) C[2]\right]\right\}\right\}$$

```
e3 = e2[[1, 1, 2, 2]]
```

$$\frac{1}{2} e^{-t} \left(1 + e^{4 t}\right) C[1] + \frac{1}{2} e^{-t} \left(-1 + e^{4 t}\right) C[2]$$

```
e5 = Expand[e3]
```

$$\frac{1}{2} e^{-t} C[1] + \frac{1}{2} e^{3 t} C[1] - \frac{1}{2} e^{-t} C[2] + \frac{1}{2} e^{3 t} C[2]$$

```
e6 = Collect[e5, e^(3 t)]
```

$$e^{-t} \left(\frac{C[1]}{2} - \frac{C[2]}{2}\right) + e^{3 t} \left(\frac{C[1]}{2} + \frac{C[2]}{2}\right)$$

$$e7 = e6 /. \left\{\left(\frac{C[1]}{2} - \frac{C[2]}{2}\right) \to c1,\ \left(\frac{C[1]}{2} + \frac{C[2]}{2}\right) \to c2\right\}$$

$$c1\, e^{-t} + c2\, e^{3 t}$$

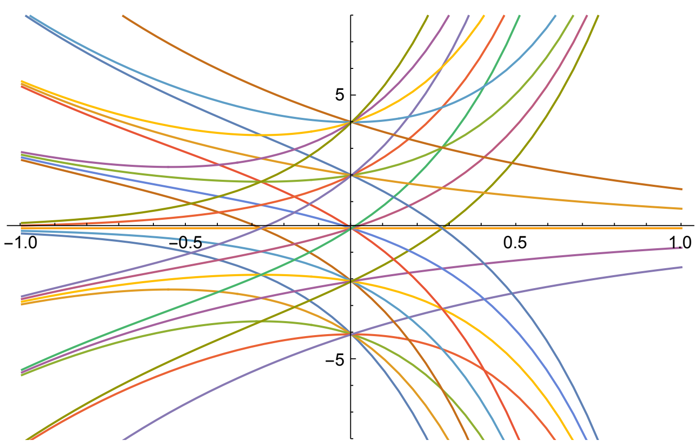
Above: y1, matching the text answer.

$$\text{Solve}\left[\left(\frac{C[1]}{2} - \frac{C[2]}{2}\right) == c1\ \&\&\ \left(\frac{C[1]}{2} + \frac{C[2]}{2}\right) == c2,\ \{c1, c2\}\right]$$

$$\left\{\left\{c1 \to \frac{1}{2} (C[1] - C[2]),\ c2 \to \frac{1}{2} (C[1] + C[2])\right\}\right\}$$

$$\text{hiy}[c1\_, c2\_, t\_] := \frac{1}{2} e^{-t} \left(1 + e^{4 t}\right) c1 + \frac{1}{2} e^{-t} \left(-1 + e^{4 t}\right) c2$$

```
plot1 =
 Plot[Evaluate[Table[hiy[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],
  {t, -1, 1}, PlotRange → {-8, 8}, PlotStyle → Thickness[0.003]]
```




```
e4 = e2[[1, 2, 2, 2]]
```

$$\frac{1}{2} e^{-t} \left(-1 + e^{4 t}\right) C[1] + \frac{1}{2} e^{-t} \left(1 + e^{4 t}\right) C[2]$$

```
e8 = Expand[e4]
```

$$-\frac{1}{2} e^{-t} C[1] + \frac{1}{2} e^{3t} C[1] + \frac{1}{2} e^{-t} C[2] + \frac{1}{2} e^{3t} C[2]$$

```
e9 = Collect[e8, e^(3 t)]
```

$$e^{-t}\left(-\frac{C[1]}{2} + \frac{C[2]}{2}\right) + e^{3t}\left(\frac{C[1]}{2} + \frac{C[2]}{2}\right)$$

$$\text{e10} = \text{e9} /. \left\{\left(-\frac{C[1]}{2} + \frac{C[2]}{2}\right) \to -\text{c1},\ \left(\frac{C[1]}{2} + \frac{C[2]}{2}\right) \to \text{c2}\right\}$$

$$-\text{c1}\, e^{-t} + \text{c2}\, e^{3t}$$

Above: y2, matching the text answer.

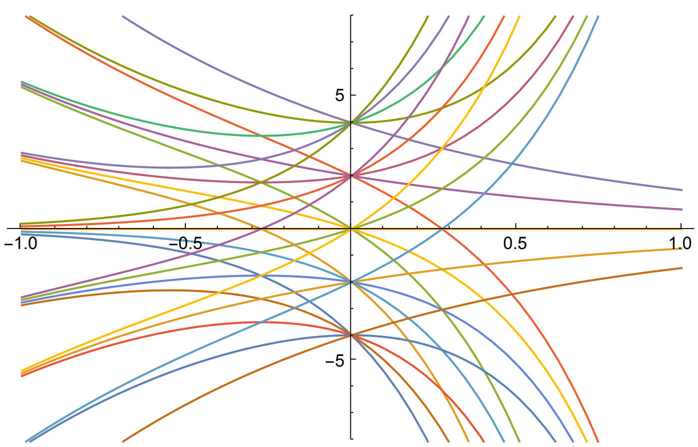
$$\text{Solve}\left[\left(-\frac{C[1]}{2} + \frac{C[2]}{2}\right) == -\text{c1}\ \&\&\ \left(\frac{C[1]}{2} + \frac{C[2]}{2}\right) == \text{c2},\ \{\text{c1}, \text{c2}\}\right]$$

$$\left\{\left\{\text{c1} \to \frac{1}{2}(C[1] - C[2]),\ \text{c2} \to \frac{1}{2}(C[1] + C[2])\right\}\right\}$$

The system of constants used in this problem is consistent with the text's, as demonstrated in pink cells.

$$\text{hiz}[\text{c1\_}, \text{c2\_}, \text{t\_}] := \frac{1}{2} e^{-t}\left(-1 + e^{4t}\right) \text{c1} + \frac{1}{2} e^{-t}\left(1 + e^{4t}\right) \text{c2}$$

```
plot2 =
 Plot[Evaluate[Table[hiz[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],
  {t, -1, 1}, PlotRange → {-8, 8}, PlotStyle → Thickness[0.003]]
```




```
Eigensystem[{{1, 2}, {2, 1}}]
```

{{3, -1}, {{1, 1}, {-1, 1}}}

```
p = 3 - 1
```

2

```
q = 3 (-1)
```

-3

```
Δ = (3 - (-1))^2
```

16

According to Table 4-1, the critical point is a saddle point. According to Table 4-2, it is unstable.

9. $y_1{}' = 4\, y_1 + y_2$
$y_2{}' = 4\, y_1 + 4\, y_2$

```
ClearAll["Global`*"]
```

```
e1 = {y1'[t] == 4 y1[t] + y2[t], y2'[t] == 4 y1[t] + 4 y2[t]}
e2 = DSolve[e1, {y1, y2}, t]
```

{y1′[t] == 4 y1[t] + y2[t], y2′[t] == 4 y1[t] + 4 y2[t]}

$$\left\{\left\{y1 \to \text{Function}\left[\{t\},\ \frac{1}{2} e^{2 t} \left(1 + e^{4 t}\right) C[1] + \frac{1}{4} e^{2 t} \left(-1 + e^{4 t}\right) C[2]\right],\ y2 \to \text{Function}\left[\{t\},\ e^{2 t} \left(-1 + e^{4 t}\right) C[1] + \frac{1}{2} e^{2 t} \left(1 + e^{4 t}\right) C[2]\right]\right\}\right\}$$

```
e3 = e2[[1, 1, 2, 2]]
```

$$\frac{1}{2} e^{2 t} \left(1 + e^{4 t}\right) C[1] + \frac{1}{4} e^{2 t} \left(-1 + e^{4 t}\right) C[2]$$

```
e4 = Expand[e3]
```

$$\frac{1}{2} e^{2 t} C[1] + \frac{1}{2} e^{6 t} C[1] - \frac{1}{4} e^{2 t} C[2] + \frac{1}{4} e^{6 t} C[2]$$

$$\text{e5} = \text{Collect}\left[\text{e4},\ e^{6 t}\right]$$

$$e^{2 t} \left(\frac{C[1]}{2} - \frac{C[2]}{4}\right) + e^{6 t} \left(\frac{C[1]}{2} + \frac{C[2]}{4}\right)$$

$$\text{e6} = \text{e5}\ /.\ \left\{\left(\frac{C[1]}{2} - \frac{C[2]}{4}\right) \to \text{c2},\ \left(\frac{C[1]}{2} + \frac{C[2]}{4}\right) \to \text{c1}\right\}$$

$$\text{c2}\ e^{2 t} + \text{c1}\ e^{6 t}$$

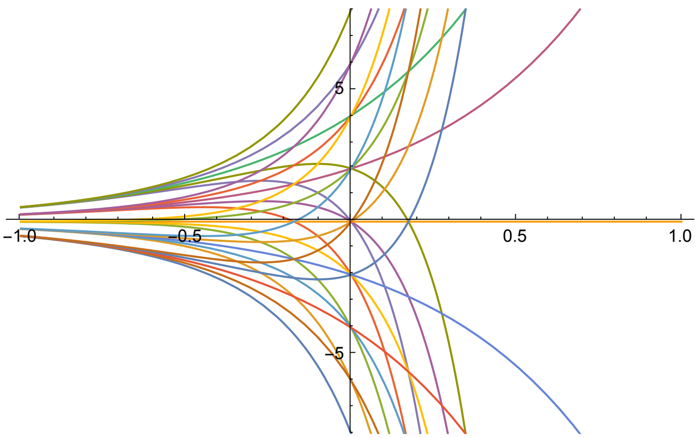
Above: the text answer for y_1.

```
Solve[(C[1]/2 - C[2]/4) == c2 && (C[1]/2 + C[2]/4) == c1, {c1, c2}]
```

$\{\{c1 \to \frac{1}{4}(2\,C[1] + C[2]),\ c2 \to \frac{1}{4}(2\,C[1] - C[2])\}\}$

```
e7[c1_, c2_, t_] := c2 e^(2 t) + c1 e^(6 t)
```

```
plot1 =
 Plot[Evaluate[Table[e7[c1, c2, t], {c1, -4, 4, 2}, {c2, -4, 4, 2}]],
  {t, -1, 1}, PlotRange → {-8, 8}, PlotStyle → Thickness[0.003]]
```




```
e8 = e2[[1, 2, 2, 2]]
```

$e^{2t}(-1 + e^{4t})\,C[1] + \frac{1}{2} e^{2t}(1 + e^{4t})\,C[2]$

```
e9 = Expand[e8]
```

$-e^{2t}\,C[1] + e^{6t}\,C[1] + \frac{1}{2} e^{2t}\,C[2] + \frac{1}{2} e^{6t}\,C[2]$

```
e10 = Collect[e9, e^(6 t)]
```

$e^{2t}\left(-C[1] + \frac{C[2]}{2}\right) + e^{6t}\left(C[1] + \frac{C[2]}{2}\right)$

```
e11 = e10 /. {(-C[1] + C[2]/2) → -2 c2, (C[1] + C[2]/2) → 2 c1}
```

$-2\,c2\,e^{2t} + 2\,c1\,e^{6t}$

Above: the text answer for y_2.

```
Solve[(-C[1] + C[2]/2) == -2 c2 && (C[1] + C[2]/2) == 2 c1, {c1, c2}]
```

```
{{c1 → 1/4 (2 C[1] + C[2]), c2 → 1/4 (2 C[1] - C[2])}}
```

```
Eigensystem[{{4, 1}, {4, 4}}]
{{6, 2}, {{1, 2}, {-1, 2}}}
```

```
p = 6 + 2
```

```
8
```

```
q = 6 × 2
```

```
12
```

```
Δ = (6 - 2)^2
```

```
16
```

According to Table 4.1, the critical point is a node. According to Table 4.2, it is unstable.

11 - 18 Trajectories of systems and second-order ODEs. Critical points.

11. Damped oscillations. Solve **y'' + 2 y' + 2 y = 0**. What kind of curves are the trajectories?

```
ClearAll["Global`*"]
```

```
eqn = y''[x] + 2 y'[x] + 2 y[x] == 0
2 y[x] + 2 y′[x] + y″[x] == 0
```

```
sol = DSolve[eqn, y, x]
```

```
{{y → Function[{x}, e^-x C[2] Cos[x] + e^-x C[1] Sin[x]]}}
```

The above green cell matches the answer in the text.

```
eqn /. sol // Simplify
{True}
```

In order to find the eigensystem, I need to make this equation into a system, using numbered lines (9) and (10) on p. 135. So I will have y_1=y, and $y_2 = y'$, and $y_3 = y''$. And the arrangement will be adopted whereby $y_1' = y_2$, and $y_2' = y_3$. Going by the text examples, the rows of the system matrix will be formed of the coefficients of the equations (lhs) of y_1' and y_2'. This will be

$y_1' = y_2$ by definition

$y'' = y_3 = y_2' = -2\,y_1 - 2\,y_2$ by problem equation description

What are the critical points? From the first expression, the first coordinate will be zero. From the second expression, the coordinates will be equal. This means that {0,0} will be the only critical point.

```
A = ( 0   1 )
    ( -2  -2 )
{{0, 1}, {-2, -2}}

{vals, vecs} = Eigensystem[A]
{{-1 + i, -1 - i}, {{-1 - i, 2}, {-1 + i, 2}}}

p = vals[[1]] + vals[[2]]
-2

q = vals[[1]] * vals[[2]]
2

Δ = (vals[[1]] - vals[[2]])^2
-4
```

According to Table 4.1, the critical point is a spiral point, and according to Table 4.2 it is stable.

17. Perturbation. The system in example 4 in section 4.3, p. 144, has a center as its critical point. Replace each a_{jk} in example 4 by a_{jk} + b. Find values of b such that you get (a) a saddle point, (b) a stable and attractive node, (c) a stable and attractive spiral, (d) an unstable spiral, (e) an unstable node.

```
In[53]:= ClearAll["Global`*"]
```

The characteristic matrix for this problem, given in the example, is like this, (but without the added ‘b’ characters).

```
In[54]:= y' = ( 0 + b    1 + b )
              ( -4 + b   0 + b )
Out[54]= {{b, 1 + b}, {-4 + b, b}}
```

I generate a table with semi-random values, but based on some rough tests.

```
In[55]:= beig =
           Table[Eigenvalues[y'], {b, {-π, -e, -2, -1.5, -1, -0.3, 0.1, 3, π, 4}}];

         Table[{{beig[[n, 1]] + beig[[n, 2]]}, {beig[[n, 1]] * beig[[n, 2]]},
             {(beig[[n, 1]] - beig[[n, 2]])^2}}, {n, 1, 10}];
```

By reviewing the characteristics of the 'beig' table entries, the qualifying seed values can be identified.

Out[134]=

n	p	q	Δ
-5.	-6.28319	-5.42478	61.1775
-4.	-5.43656	-4.15485	46.1756
-3.	-4.	-2.	24.
-2.	-3.	-0.5	11.
-1.	-2.	1.	0.
0.	-0.6 + 0. i	3.1 + 0. i	-12.04 + 0. i
1.	0.2 + 0. i	4.3 + 0. i	-17.16 + 0. i
2.	6.	13.	-16.
3.	6.28319 + 0. i	13.4248 + 0. i	-14.2207 + 0. i
4.	8.	16.	0.

The grid below identifies the 'n' number critical points contained in the **Grid** above with the required characteristics, based on Table 4.1 and 4.2.

Out[150]=

Conforming n	Features
-3	unstable saddle point
-1	stable and attrac node
0	stable and attrac spiral
2	unstable spiral
4	unstable node